Dividing Polynomials

Long Division (The Old Fashioned Way)

Example:

$$\frac{6x^2 - 26x + 12}{x - 4}$$

$$x-4\,\overline{)\,6x^2 - 26x + 12}$$

We see that x goes into $6x^2$ $6x$ times so we multiply and subtract

$$\begin{array}{r@{}l}
 & \underline{\;6x\qquad\qquad\;} \\
x-4) & 6x^2 - 26x + 12 \\
 & \underline{6x^2 - 24x} \\
 & \qquad\quad 2x + 12
\end{array}$$

Now we see that x goes into $2x$ 2 times

$$\begin{array}{r@{}l}
 & \underline{\;6x \;\;+\;\; 2\qquad} \\
x-4) & 6x^2 - 26x + 12 \\
 & \underline{6x^2 - 24x} \\
 & \qquad\quad 2x + 12 \\
 & \qquad\quad \underline{2x - \;\;8} \\
 & \qquad\qquad\quad 20
\end{array}$$

So we find $\dfrac{6x^2 - 26x + 12}{x - 4} = 6x + 2 + \dfrac{20}{x - 4}$

Note that if you are missing terms, you should include them.
Example:

$$\frac{8x^4 + 6x^2 - 3x + 1}{2x^2 - x + 2}$$

$$2x^2 - x + 2 \overline{)8x^4 + 0x^3 + 6x^2 - 3x + 1}$$

Synthetic Division

Synthetic division is a somewhat easier method if done right however it only works if you are dividing by a first degree polynomial of the form

x-c

Example:

$$\frac{2x^3-7x^2+5}{x-3}$$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline & & 6 & & \\ \hline & 2 & -1 & & \end{array}$$

(Add

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline & & 6 & -3 & \\ \hline & 2 & -1 & -3 & \end{array}$$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -4 \end{array}$$

So:

$$\frac{2x^3-7x^2+5}{x-3}=2x^2-x-3+\frac{-4}{x-3}$$

or

$$2x^3-7x^2+5=(x-3)(2x^2-x-3)-4$$

To understand why synthetic division works we need to understand:

The Remainder Theorem

The remainder when dividing a polynomial $P(x)$ by (x-c) is $P(c)$.

Proof

If we divide a polynomial $P(x)$ by (x-c) we get a lower degree polynomial Q and a remainder which is a number.

$\frac{P(x)}{x-c} = Q(x) + \frac{r}{x-c}$ where $Q(x)$ is a polynomial of degree less than $P(x)$ and r is a constant.

We can rewrite this as:

$$P(x) = (x-c)Q(x) + r$$

But then if we plug in c

$$P(c) = (c-c)Q(c) + r = r$$

so

$$P(c) = r$$

Example:

$$P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$$

Find $P(x)/(x+2)$

Solution: $3x^4 - x^3 - 2x^2 + 4x - 1$ Remainder 5

Find $P(-2)$

Since we just divided $P(x)$ by $\left(x - {}^{-}2\right)$ $P(-2) = 5$

The Factor Theorem

A corollary to the remainder theorem is the factor theorem.

c is a zero of *P* if and only if *x*-*c* is a factor of *P*(*x*).

Because this is an if and only if statement, there are two theorems and therefore two different proofs needed.

First, by the remainder theorem

$$P(x) = Q(x)(x - c) + r$$

If *c* is a zero of *P* then

$$P(c) = Q(c)(c - c) + r$$
$$P(c) = r = 0$$

So if *c* is a zero of *P* then *r* is zero.
But that means $P(x) = Q(x)(x - c)$
which means that *x*-*c* is a factor of *P*(*x*).

On the other hand, what if *x*-*c* is a factor of *P*(*x*).

Then $P(x) = Q(x)(x - c)$

but then $P(c) = Q(c)(c - c) = 0$
so *c* is a zero of *P*.

Let's look at our synthetic division example again.

$$\frac{2x^3-7x^2+5}{x-3}$$

First rewrite the polynomial we are going to divide into as follows

$$2x^3-7x^2+0x+5$$
$$x\left(2x^2-7x+0\right)+5$$
$$x\left(x\left(2x-7\right)+0\right)+5$$

Note that now 2 will be the coefficient of x^2 after the division.

Now plug in the value 3 to this and see what happens

$$x\left(x\left(2\cdot 3-7\right)+0\right)+5$$

So our first step is $2\cdot 3$ and then we add 6 to -7 leaving -1

Note that now -1 will be the coefficient of x after the division.

$$x\left(3\cdot -1+0\right)+5$$

Then we multiply 3 by -1 and add it to 0 leaving -3

Note that now -3 is the coefficient of x^0 after the division.

Finally we multiply 3 by -3 and add it to 5 leaving -4 the remainder

$$3\left(-3\right)+5$$

Notice that these are the exact same calculation steps that we took with synthetic division.

The process of synthetic division is the same as computing $P\left(c\right)$, which by the remainder theorem will give us the remainder of multiplying $P\left(c\right)$ by c, and in the process we retain the coefficients of the quotient.

Factoring using the Factor Theorem

Factor $f(x) = x^3 - 7x + 6$

We first have to guess a zero.

Note that $f(1) = 1 - 7 + 6 = 0$ so (x-1) is a factor of f because the remainder is zero.

$$\frac{x^3 - 7x + 6}{x - 1} = x^2 + x - 6 = (x+3)(x-2)$$

so

$$f(x) = (x-1)(x-2)(x+3)$$